

Tamed EM scheme of Neutral Stochastic Differential Delay Equations

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Abstract

In this paper, we investigate the convergence of the tamed Euler-Maruyama (EM) scheme for a class of neutral stochastic differential delay equations. The strong convergence results of the tamed EM scheme are presented under global and local non-Lipschitz conditions, respectively. Moreover, under a global Lipschitz condition, we provide the rate of the convergence of tamed EM, which is smaller than the rate convergence of classical EM scheme one half.

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1 Introduction

The Euler-Maruyama scheme is of vital importance in numerical approximation for stochastic differential equations (SDEs). In [18], Kloeden and Platen illustrated that, if the coefficients of an SDE are globally Lipschitz continuous, then the EM approximation converges to the exact solution of the SDE in both strong and weak sense, the convergence rates for both cases are provided as well. In the same book, they also mentioned that the Milstein scheme converges to the exact solution of SDE in both strong and weak sense with different orders under certain conditions including the global Lipschitz condition. It is the first time that Higham, Mao and Stuart [10] established strong convergence results under the super-linear condition and the moment boundedness condition, however, it remained an open question whether the moment of the EM approximation is bounded within finite time if the coefficients of an SDE are not globally Lipschitz continuous. Recently, Hutzenhaler, Jentzen and Kloeden [8] have found that once the global Lipschitz condition was replaced by the super-linear condition, the moment of the EM scheme could be infinity within the finite time. To tackle this problem, in the paper [9], Hutzenhaler, Jentzen and Kloeden introduced a new approximation scheme, which is the so-call tamed EM scheme. By employing the tamed scheme, the drift coefficient is tamed so that it is uniformly

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bounded. With such an approach, it is proved that the tamed EM scheme converges to the exact solution of the SDE under the super-linear condition of the drift coefficients. The tamed EM scheme is later extended to SDEs with locally Lipschitz condition of the diffusion by Sabanis [26].

On the other hand, stochastic differential delay equations (SDDEs) and neutral stochastic differential delay equations (NSDDEs) describe a wide variety of natural and man-made systems. For the theories and applications of SDDEs and NSDDEs, we here only mention [1, 2, 4, 5, 6, 7, 13, 15, 19, 23, 24, 25], to name a few. Since most SDDEs and NSDDEs can not be solved explicitly, numerical methods have become essential. Recently, an extensive literature has emerged in investigating the strong convergence, weak convergence or sample path convergence of numerical schemes for SDDEs and NSDDEs, for example, [3, 11, 12, 15, 17, 22]. We should point out that the strong convergence of EM schemes for SDDEs is, in general, discussed under a linear growth condition or bounded moments of analytic and numerical solutions, e.g., [15, 17, 22]. However, similar to the SDEs case, it remained an open question whether the EM scheme converges to the exact solution if the coefficients of the SDDEs and NSDDEs are under the super-linear condition. The main aim of this paper is to answer this question by extending the tamed EM method to NSDDEs. Because of the neutral term, the technical details increased significantly.

The rest of this paper will be organised as follows. In section 2, some notation and preliminaries are introduced. In section 3, p -th moment boundedness, convergence of EM scheme under global and local monotonicity conditions are proven, moreover, the rate of convergence is provided under a global conditions. In section 4, we present similar results as in section 3 while the Brownian motion is replaced by the pure jump processes.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (i.e. it is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets). Let $\tau > 0$ be a constant and denote $C([-\tau, 0]; \mathbb{R}^n)$ the space of all continuous functions from $[-\tau, 0]$ to \mathbb{R}^n with the norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Let $B(t)$ be a standard m -dimensional Brownian motion.

Consider an n -dimensional neutral stochastic differential delay equation

$$d[X(t) - D(X(t - \tau))] = b(X(t), X(t - \tau))dt + \sigma(X(t), X(t - \tau))dB(t), \quad (2.1)$$

on $t \geq 0$, where

$$D : \mathbb{R}^n \rightarrow \mathbb{R}^n, b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m},$$

we assume that D , b and σ are Borel-measurable and the initial data satisfies the following condition: for any $p \geq 2$

$$\{X(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n), \quad (2.2)$$

that is ξ is an \mathcal{F}_0 -measurable $C([-\tau, 0]; \mathbb{R}^n)$ -valued random variable and $E\|\xi\|^p < \infty$. Now, fix $T > \tau > 0$, without loss of generality, we assume that T and τ are rational numbers, and the step size $h \in (0, 1)$ be fraction of τ and T , so that there exist two positive integers

M, \bar{M} such that $h = T/M = \tau/\bar{M}$. Throughout the paper, unless otherwise stated, $C > 0$ denotes a generic constant, independent of h , whose value may change from line to line.

For the future use, we assume that:

(A1) There exists a positive constant \tilde{K} such that

$$\langle x - D(y), b(x, y) \rangle \vee \|\sigma(x, y)\|^2 \leq \tilde{K}(1 + |x|^2 + |y|^2), \quad (2.3)$$

for $\forall x, y \in \mathbb{R}^n$.

(A2) $D(0) = 0$ and there exists a constant $\kappa \in (0, 1)$ such that

$$|D(x) - D(\bar{x})| \leq \kappa|x - \bar{x}| \text{ for all } x, \bar{x} \in \mathbb{R}^n. \quad (2.4)$$

(A3) For any $R > 0$, there exist two positive constants \tilde{K}_R and K_R such that

$$\begin{aligned} & \langle x - D(y) - \bar{x} + D(\bar{y}), b(x, y) - b(\bar{x}, \bar{y}) \rangle \vee \|\sigma(x, y) - \sigma(\bar{x}, \bar{y})\|^2 \\ & \leq \tilde{K}_R(|x - \bar{x}|^2 + |y - \bar{y}|^2), \end{aligned} \quad (2.5)$$

for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$,
and

$$\sup_{|x| \vee |y| \leq R} |b(x, y)| \leq K_R. \quad (2.6)$$

(A4) There exist two positive constants l and L such that,

$$\langle x - D(y) - \bar{x} + D(\bar{y}), b(x, y) - b(\bar{x}, \bar{y}) \rangle + \|\sigma(x, y) - \sigma(\bar{x}, \bar{y})\|^2 \leq L(|x - \bar{x}|^2 + |y - \bar{y}|^2),$$

and

$$|b(x, y) - b(\bar{x}, \bar{y})| \leq L(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x - \bar{x}| + |y - \bar{y}|),$$

for all x, y, \bar{x} and $\bar{y} \in \mathbb{R}^n$.

(A5) For any $s, t \in [-\tau, 0]$ and $q > 0$, let $\mathbb{E}\|\xi(t) - \xi(s)\|^q \leq L|t - s|^q$.

Remark 2.1 Assume that **(A1)**-(**A3**) hold, then NSDDE (2.1) with initial (2.2) admits a unique strong global solution $X(t)$, $t \in [0, T]$. The proof details of such existence and uniqueness result can be found in [16]. If assumption **(A3)** is replaced by **(A4)**, the theorem of existence and uniqueness still holds.

We now define

$$b_h(x, y) := \frac{b(x, y)}{1 + h^\alpha |b(x, y)|}, \quad (2.7)$$

for all $x, y \in \mathbb{R}^n$ and $\alpha \in (0, \frac{1}{2}]$. The reason why such value of α is chosen will be revealed in Section 3. By observation, one has

$$|b_h(x, y)| \leq \min(h^{-\alpha}, |b(x, y)|).$$

Remark 2.2 It is easy to verify that $b_h(x, y) = \frac{b(x, y)}{1 + h^\alpha |b(x, y)|}$ satisfies all assumptions **(A1)**, **(A3)** and **(A4)**. For the assumption **(A1)**, we have

$$\begin{aligned} 2\langle x - D(y), b_h(x, y) \rangle &= \frac{2}{1 + h^\alpha |b(x, y)|} \langle x - D(y), b(x, y) \rangle \\ &\leq \frac{\tilde{K}}{1 + h^\alpha |b(x, y)|} (1 + |x|^2 + |y|^2) \\ &\leq \tilde{K} (1 + |x|^2 + |y|^2). \end{aligned}$$

For the assumption **(A3)**, which is a local assumption, we can derive that

$$\begin{aligned} 2\langle x - D(y) - \bar{x} + D(\bar{y}), b_h(x, y) - b_h(\bar{x}, \bar{y}) \rangle \\ \leq \bar{K}_R (|x - \bar{x}|^2 + |y - \bar{y}|^2), \end{aligned}$$

where $\bar{K}_R = \frac{\tilde{K}_R}{1 + h^\alpha (|b(\bar{x}, \bar{y})| \wedge |b(x, y)|)}$. The verification detail for assumption **(A4)** is omitted, since it is similar to **(A3)**.

Now, we can define the discrete-time tamed EM scheme:

For every integer $n = -\bar{M}, \dots, 0$, $Y_h^{(n)} = \xi(nh)$. For every integer $n = 1, \dots, M - 1$,

$$Y_h^{(n+1)} - D(Y_h^{(n+1-\bar{M})}) = Y_h^{(n)} - D(Y_h^{(n-\bar{M})}) + b_h(Y_h^{(n)}, Y_h^{(n-\bar{M})})h + \sigma(Y_h^{(n)}, Y_h^{(n-\bar{M})})\Delta B_h^{(n)}, \quad (2.8)$$

where $\Delta B_h^{(n)} = B((n+1)h) - B(nh)$. The discrete-time tamed EM scheme (2.8) can be rewritten as

$$\begin{aligned} Y_h^{(n+1)} &= D(Y_h^{(n+1-\bar{M})}) + \xi(0) - D(\xi(-\tau)) + \sum_{i=1}^n b_h(Y_h^{(i)}, Y_h^{(i-\bar{M})})h \\ &\quad + \sum_{i=1}^n \sigma(Y_h^{(i)}, Y_h^{(i-\bar{M})})\Delta B_h^{(i)}. \end{aligned} \quad (2.9)$$

For $t \in [nh, (n+1)h]$, we set $\bar{Y}(t) := Y_h^{(n)}$. Since $\tau = \bar{M}h$, $\bar{Y}(t - \tau) = Y_h^{(n-\bar{M})}$. For the sake of simplicity, we define the corresponding continuous-time tamed EM approximate solution $Y(t)$ as follows: For any $\theta \in [-\tau, 0]$, $Y(\theta) = \xi(\theta)$. For any $t \in [0, T]$,

$$\begin{aligned} Y(t) &= D(\bar{Y}(t - \tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^t b_h(\bar{Y}(s), \bar{Y}(s - \tau))ds \\ &\quad + \int_0^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau))dB(s). \end{aligned} \quad (2.10)$$

Noting that for any $t \in [0, T]$, there exists a positive integer n , $0 \leq n \leq M - 1$, such that $nh \leq t < (n+1)h$, we have

$$\begin{aligned} Y(t) &= D(\bar{Y}(t - \tau)) + \xi(0) - D(\xi(-\tau)) + \int_0^{nh} b_h(\bar{Y}(s), \bar{Y}(s - \tau))ds \\ &\quad + \int_0^{nh} \sigma(\bar{Y}(s), \bar{Y}(s - \tau))dB(s) + \int_{nh}^t b_h(\bar{Y}(s), \bar{Y}(s - \tau))ds \\ &\quad + \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau))dB(s) \\ &= Y(nh) + \int_{nh}^t b_h(\bar{Y}(s), \bar{Y}(s - \tau))ds + \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau))dB(s). \end{aligned} \quad (2.11)$$

This means the continuous-time tamed EM approximate solution $Y(t)$ coincides with the discrete-time tamed approximation solution $\bar{Y}(t)$ at grid points $t = nh$, $n = 0, 1, \dots, M-1$.

3 Tamed EM Method of NSDDEs driven by Brownian Motion

In this section, we show that the tamed EM scheme converges to the exact solution under certain conditions, i.e. we have the following main results:

Theorem 3.1 *Suppose that (A1), (A2), (A4) and (A5) hold, then the tamed EM scheme (2.10) converges to the exact solution of (2.1) such that for any $p \geq 2$,*

$$E \left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right] \leq Ch^{\alpha p}. \quad (3.1)$$

The next theorem states that the convergence result still holds if the global monotonic condition (A4) is replaced by its local counterpart (A3). However, we are unable to provide the convergence rate under this weaker condition:

Theorem 3.2 *Suppose (A1)-(A3) and (A5) hold, then the tamed EM scheme (2.10) converges to the exact solution of (2.1) such that for any $p \geq 2$,*

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right] = 0. \quad (3.2)$$

3.1 Moment Bounds

In order to prove our main results, in this subsection we investigate the boundedness of moments of both exact solution and EM approximation.

Lemma 3.1 *Consider the continuous-time tamed EM scheme given by equation (2.11). If for some $p \geq 2$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}(|Y(t)|^p) \leq C, \quad (3.3)$$

and (A1) hold, then it holds that

$$\mathbb{E} \left[\sup_{0 \leq n \leq M-1} \sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p \right] \leq Ch^{p/2}, \quad (3.4)$$

and

$$\mathbb{E} \left[\sup_{0 \leq n \leq M-1} \sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p |b_h(\bar{Y}(t), \bar{Y}(t - \tau))|^p \right] \leq C. \quad (3.5)$$

Proof: By the definition of the tamed EM scheme, we have for $nh \leq t < (n+1)h$,

$$\begin{aligned} \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p \right] &= \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t b_h(\bar{Y}(s), \bar{Y}(s - \tau)) ds \right. \right. \\ &\quad \left. \left. + \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \right|^p \right]. \end{aligned}$$

Therefore, due to Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p \right] &\leq 2^{p-1} h^{p-1} \mathbb{E} \left[\int_{nh}^{(n+1)h} \left| b_h(\bar{Y}(s), \bar{Y}(s-\tau)) \right|^p ds \right] \\ &\quad + 2^{p-1} \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s-\tau)) dB(s) \right|^p \right]. \end{aligned} \quad (3.6)$$

Using the Burkholder-Davis-Gundy(BDG) inequality [20, Theorem 1.7.3 page 40] and (3.3), for some $p \geq 2$, we can derive that for $nh \leq t < (n+1)h$,

$$\begin{aligned} &\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t \sigma(\bar{Y}(s), \bar{Y}(s-\tau)) dB(s) \right|^p \right] \\ &\leq C \mathbb{E} \left[\int_{nh}^{(n+1)h} \|\sigma(\bar{Y}(s), \bar{Y}(s-\tau))\|^2 ds \right]^{p/2} \\ &\leq C \mathbb{E} \left[\int_{nh}^{(n+1)h} (1 + |\bar{Y}(s)|^2 + |\bar{Y}(s-\tau)|^2) ds \right]^{p/2} \\ &\leq Ch^{p/2}. \end{aligned}$$

This, together with $|b_h(\bar{Y}(s), \bar{Y}(s-\tau))| \leq h^{-\alpha}$, yields

$$\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p \right] \leq 2^{p-1} h^{(1-\alpha)p} + Ch^{p/2} \leq Ch^{p/2}. \quad (3.7)$$

Therefore (3.4) holds. Moreover,

$$\begin{aligned} &\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p |b_h(\bar{Y}(s), \bar{Y}(s-\tau))|^p \right] \\ &\leq \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |Y(t) - Y(nh)|^p \right] h^{-\alpha p} \\ &\leq Ch^{(1/2-\alpha)p} \leq C. \end{aligned} \quad (3.8)$$

In (3.7) and (3.8), we have used the fact that $\alpha \in (0, 1/2]$. The proof is therefore complete. \square

Lemma 3.2 *Assume that (A1), (A2) and (A5) hold. Then there exists a positive constant C independent of h such that for any $p \geq 2$,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^p \right] \vee \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y(t)|^p \right] \leq C. \quad (3.9)$$

Proof. By [20, Lemma 4.4, p212] and assumption (A2), we know that for any $p \geq 2$,

$$\sup_{0 \leq t \leq T} |X(t)|^p \leq \frac{\kappa}{1-\kappa} \|\xi\|^p + \frac{1}{(1-\kappa)^p} \sup_{0 \leq t \leq T} |X(t) - D(X(t-\tau))|^p, \quad (3.10)$$

An application of Itô's formula yields

$$\begin{aligned}
& |X(t) - D(X(t - \tau))|^p \leq |\xi(0) - D(\xi(-\tau))|^p \\
& + p \int_0^t |X(t) - D(X(t - \tau))|^{p-2} \langle X(s) - D(X(s - \tau)), b(X(s), X(s - \tau)) \rangle ds \\
& + \frac{p(p-1)}{2} \int_0^t |X(t) - D(X(t - \tau))|^{p-2} \|\sigma(X(s), X(s - \tau))\|^2 ds \\
& + p \int_0^t |X(t) - D(X(t - \tau))|^{p-2} (X(s) - D(X(s - \tau)))^T (\sigma(X(s), X(s - \tau))) dB(s) \\
& =: |\xi(0) - D(\xi(-\tau))|^p + H_1(t) + H_2(t) + H_3(t).
\end{aligned} \tag{3.11}$$

Under **(A1)** and **(A2)**, one has

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} H_1(t)) + \mathbb{E}(\sup_{0 \leq t \leq T} H_2(t)) \\
& \leq C \mathbb{E} \int_0^T |X(s) - D(X(s - \tau))|^{p-2} (1 + |X(s)|^2 + |X(s - \tau)|^2) ds \\
& \leq C \mathbb{E} \int_0^T (|X(s)|^{p-2} + |D(X(s - \tau))|^{p-2}) (1 + |X(s)|^2 + |X(s - \tau)|^2) ds \\
& \leq C \mathbb{E} \int_0^T (1 + |X(s)|^p + |X(s - \tau)|^p) ds \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u)|^p) ds.
\end{aligned} \tag{3.12}$$

By the Burkholder-Davis-Gundy(BDG) inequality and the Young inequality, we derive that

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} H_3(t)) \leq C \mathbb{E} \left(\int_0^T (|X(s) - D(X(s - \tau))|^{2p-2} \|\sigma(X(s), X(s - \tau))\|^2) ds \right)^{1/2} \\
& \leq C \mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^{p-1} \left(\int_0^T \|\sigma(X(s), X(s - \tau))\|^2 ds \right)^{1/2} \right) \\
& \leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^p) + C \left(\int_0^T \|\sigma(X(s), X(s - \tau))\|^2 ds \right)^{p/2} \\
& \leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^p) + C \left(\int_0^T (1 + |X(s)|^2 + |X(s - \tau)|^2) ds \right)^{p/2} \\
& \leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^p) + C \int_0^T \mathbb{E}(1 + |X(s)|^p + |X(s - \tau)|^p) ds \\
& \leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^p) + C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u)|^p) ds.
\end{aligned} \tag{3.13}$$

Now, substituting (3.12) and (3.13) into (3.11), we then have

$$\mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau))|^p) \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u)|^p) ds. \tag{3.14}$$

This, together with (3.10), implies that

$$\mathbb{E}(\sup_{0 \leq t \leq T} |X(t)|^p) \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u)|^p) ds.$$

The desired assertion for the exact solution follows an application of Gronwall's inequality.

In order to estimate the p -th moment of the tamed EM scheme (2.10), an inductive argument is used below. Firstly, we claim that there exists a constant C such that:

$$\sup_{0 \leq t \leq T} \mathbb{E}|Y(t)|^2 \leq C. \quad (3.15)$$

An application of the Itô formula yields,

$$\begin{aligned} & |Y(t) - D(\bar{Y}(t - \tau))|^2 \\ &= |\xi(0) - D(\xi(-\tau))|^2 + 2 \int_0^t \langle Y(s) - D(\bar{Y}(s - \tau)), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\ &+ \int_0^t \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds + 2 \int_0^t \langle Y(s) - D(\bar{Y}(s - \tau)), \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \rangle \\ &= |\xi(0) - D(\xi(-\tau))|^2 + 2 \int_0^t \langle \bar{Y}(s) - D(\bar{Y}(s - \tau)), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\ &+ 2 \int_{t_0}^t \langle Y(s) - \bar{Y}(s), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds + \int_0^t \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \\ &+ 2 \int_0^t \langle Y(s) - D(\bar{Y}(s - \tau)), \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \rangle \\ &=: |\xi(0) - D(\xi(-\tau))|^2 + \hat{H}_1(t) + \hat{H}_2(t) + \hat{H}_3(t) + \hat{H}_4(t). \end{aligned} \quad (3.16)$$

By **(A1)** and **(A2)**, we compute

$$\begin{aligned} \mathbb{E}(\hat{H}_1(t)) + \mathbb{E}(\hat{H}_3(t)) &\leq C \mathbb{E} \int_0^t 1 + |\bar{Y}(s)|^2 + |\bar{Y}(s - \tau)|^2 ds \\ &\leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}(|Y(u)|^2) ds. \end{aligned} \quad (3.17)$$

Using the definition of $Y(s)$ and $\bar{Y}(s)$, we have

$$\begin{aligned} \mathbb{E}(\hat{H}_2(t)) &= 2 \mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s b_h(\bar{Y}(r), \bar{Y}(r - \tau)) dr, b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \right\rangle ds \\ &+ 2 \mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s \sigma(\bar{Y}(r), \bar{Y}(r - \tau)) dB(r), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \right\rangle ds \\ &= 2 \mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s b_h(\bar{Y}(r), \bar{Y}(r - \tau)) dr, b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \right\rangle ds \\ &+ 2 \mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s \sigma(\bar{Y}(r), \bar{Y}(r - \tau)) dB(r) | \mathcal{F}_{[\frac{s}{h}]h}, b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \right\rangle ds \\ &= 2 \mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s b_h(\bar{Y}(r), \bar{Y}(r - \tau)) dr, b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \right\rangle ds \\ &\leq C t h^{1-2\alpha} \leq C, \end{aligned} \quad (3.18)$$

here again we have used $\alpha \in (0, 1/2]$. Using the Young inequality, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E}(|Y(t)|^2) &= \sup_{0 \leq t \leq T} \mathbb{E}(|Y(t) - D(\bar{Y}(t - \tau)) + D(\bar{Y}(t - \tau))|^2) \\ &\leq C + \frac{1}{2} \sup_{0 \leq t \leq T} \mathbb{E}(|Y(t)|^2) + C \sup_{0 \leq t \leq T} \mathbb{E}(|Y(t) - D(\bar{Y}(t - \tau))|^2), \end{aligned} \quad (3.19)$$

Now, substituting (3.17) and (3.18) into (3.19), we can derive that

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E}(|Y(t)|^2) &\leq C + C \left(\sup_{0 \leq t \leq T} \mathbb{E}(|Y(t) - D(\bar{Y}(t - \tau))|^2) \right) \\ &\leq C + C \int_0^T \sup_{0 \leq u \leq s} \mathbb{E}(|Y(u)|^2) ds. \end{aligned}$$

The assertion (3.15) follows an application of the Gronwall inequality.

In the sequel, we are now going to show that there exists a constant $C > 0$ such that for any $p \geq 2$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y(t)|^p \right] \leq C. \quad (3.20)$$

Letting $p = 4$ and using the Itô formula, we have

$$\begin{aligned} |Y(t) - D(\bar{Y}(t - \tau))|^p &\leq |\xi(0) - D(\xi(-\tau))|^p + p \int_0^t |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \\ &\quad \times \langle \bar{Y}(s) - D(\bar{Y}(s - \tau)), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\ &\quad + p \int_0^t |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \langle Y(s) - \bar{Y}(s), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\ &\quad + \frac{p(p-1)}{2} \int_0^t |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \\ &\quad + p \int_0^t |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \langle Y(s) - D(\bar{Y}(s - \tau)), \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \rangle \\ &=: |\xi(0) - D(\xi(-\tau))|^p + \bar{H}_1(t) + \bar{H}_2(t) + \bar{H}_3(t) + \bar{H}_4(t). \end{aligned} \quad (3.21)$$

Under assumptions **(A1)** and **(A2)**, there exists a constant $C > 0$ such that

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq t \leq T} \bar{H}_1(t) \right) + \mathbb{E} \left(\sup_{0 \leq t \leq T} \bar{H}_3(t) \right) \\ &\leq C \mathbb{E} \int_0^T |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} (1 + |\bar{Y}(s)|^2 + |\bar{Y}(s - \tau)|^2) ds \\ &\leq C \mathbb{E} \int_0^T (|Y(s)|^{p-2} + |D(\bar{Y}(s - \tau))|^{p-2}) (1 + |\bar{Y}(s)|^2 + |\bar{Y}(s - \tau)|^2) ds \\ &\leq C \mathbb{E} \int_0^T (1 + |Y(s)|^p + |\bar{Y}(s)|^p + |\bar{Y}(s - \tau)|^p) ds \leq C + C \int_0^T \mathbb{E} \left(\sup_{0 \leq u \leq s} |Y(u)|^p \right) ds. \end{aligned} \quad (3.22)$$

Using Young's inequality, we derive that

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq t \leq T} \bar{H}_2(t) \right) \\ &\leq C \mathbb{E} \int_0^T |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \langle Y(s) - \bar{Y}(s), b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\ &\leq C \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^{p-2} \times \left[\left(\int_0^T |Y(s) - \bar{Y}(s)| |b_h(\bar{Y}(s), \bar{Y}(s - \tau))| ds \right)^{1/2} \right]^2 \right) \\ &\leq \frac{1}{4} \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^p + C \mathbb{E} \int_0^T |Y(s) - \bar{Y}(s)|^{p/2} |b_h(\bar{Y}(s), \bar{Y}(s - \tau))|^{p/2} ds \\ &\leq \frac{1}{4} \mathbb{E} \sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^p + C, \end{aligned} \quad (3.23)$$

where we have applied result from obtained Lemma 3.1 to the second term above.

By the BDG inequality again, we have

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \bar{H}_4(t)) \\
& \mathbb{E}(\sup_{0 \leq t \leq T} p \int_0^t |Y(s) - D(\bar{Y}(s - \tau))|^{p-2} \langle Y(s) - D(\bar{Y}(s - \tau)), \sigma(\bar{Y}(s), \bar{Y}(s - \tau)) dB(s) \rangle) \\
& \leq C \mathbb{E} \left(\int_0^T (|Y(s) - D(\bar{Y}(s - \tau))|^{2p-2} \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2) ds \right)^{1/2} \\
& \leq C \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^{p-1} \left(\int_0^T \|\sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \right)^{1/2} \right) \\
& \leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^p) + C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |Y(u)|^p) ds.
\end{aligned} \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.21) and using (3.10), for $p = 4$, we can derive that

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} |Y(t)|^p) \leq C + C \mathbb{E}(\sup_{0 \leq t \leq T} |Y(t) - D(\bar{Y}(t - \tau))|^p) \\
& \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |Y(u)|^p) ds.
\end{aligned}$$

The required assertion follows from an application of the Grownwall inequality. By repeating the same procedure, the desired result (3.9) can be obtained by induction. \square

3.2 Proof of Main Results

In this subsection, we give proofs for the main results.

Proof of Theorem 3.1: For any $0 \leq t \leq T$, by [20, Lemma 6.4.1] as well as the assumption **(A2)**, we have, for $p \geq 2$ and $\varepsilon > 0$,

$$\begin{aligned}
& |X(t) - Y(t)|^p = |X(t) - Y(t) - D(X(t - \tau)) + D(\bar{Y}(t - \tau)) \\
& \quad + D(X(t - \tau)) - D(\bar{Y}(t - \tau))|^p \\
& \leq \left[1 + \varepsilon^{\frac{1}{p-1}} \right]^{p-1} \left(\frac{|D(X(t - \tau)) - D(\bar{Y}(t - \tau))|^p}{\varepsilon} \right. \\
& \quad \left. + |X(t) - D(X(t - \tau)) - Y(t) + D(\bar{Y}(t - \tau))|^p \right) \\
& \leq \left[1 + \varepsilon^{\frac{1}{p-1}} \right]^{p-1} \left(\frac{\kappa^p |X(t - \tau) - \bar{Y}(t - \tau)|^p}{\varepsilon} \right. \\
& \quad \left. + |X(t) - D(X(t - \tau)) - Y(t) + D(\bar{Y}(t - \tau))|^p \right).
\end{aligned} \tag{3.25}$$

Letting $\varepsilon = \lfloor \frac{\kappa}{1-\kappa} \rfloor^{p-1}$, (3.25) yields that

$$\begin{aligned}
|X(t) - Y(t)|^p &\leq \kappa |X(t - \tau) - \bar{Y}(t - \tau)|^p \\
&\quad + \frac{1}{(1 - \kappa)^{p-1}} |X(t) - D(X(t - \tau)) - Y(t) + D(\bar{Y}(t - \tau))|^p \\
&\leq \kappa |X(t) - \bar{Y}(t)|^p I_{[-\tau \leq t \leq 0]}(t) + \kappa |X(t) - \bar{Y}(t)|^p I_{[0 \leq t \leq T]}(t) \\
&\quad + \frac{1}{(1 - \kappa)^{p-1}} |X(t) - D(X(t - \tau)) - Y(t) + D(\bar{Y}(t - \tau))|^p.
\end{aligned} \tag{3.26}$$

This, together with **(A5)**, implies

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right) &\leq \frac{1}{(1 - \kappa)^p} \mathbb{E} \left(\sup_{0 \leq t \leq T} |X(t) - D(X(t - \tau)) \right. \\
&\quad \left. - Y(t) + D(\bar{Y}(t - \tau))|^p \right) + Ch^p.
\end{aligned} \tag{3.27}$$

By Itô's formula we can show that

$$\begin{aligned}
&|X(t) - D(X(t - \tau)) - Y(t) + D(\bar{Y}(t - \tau))|^p \\
&\leq p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), b(X(s), X(s - \tau)) - b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\
&\quad + \frac{p(p-1)}{2} \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \|\sigma(X(s), X(s - \tau)) - \sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \\
&\quad + p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), \\
&\quad \quad (\sigma(X(s), X(s - \tau)) - \sigma(\bar{Y}(s), \bar{Y}(s - \tau))) dB(s) \rangle \\
&= p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), b(X(s), X(s - \tau)) - b(Y(s), \bar{Y}(s - \tau)) \rangle ds \\
&\quad + p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), b(Y(s), \bar{Y}(s - \tau)) - b(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\
&\quad + p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), b(\bar{Y}(s), \bar{Y}(s - \tau)) - b_h(\bar{Y}(s), \bar{Y}(s - \tau)) \rangle ds \\
&\quad + \frac{p(p-1)}{2} \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \|\sigma(X(s), X(s - \tau)) - \sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \\
&\quad + p \int_0^t |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times \langle X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau)), \\
&\quad \quad (\sigma(X(s), X(s - \tau)) - \sigma(\bar{Y}(s), \bar{Y}(s - \tau))) dB(s) \rangle \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t).
\end{aligned} \tag{3.28}$$

By assumption **(A2)** and **(A4)**, we have

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} (I_1(t))) &\leq C \mathbb{E} \int_0^T |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-2} \\
&\quad \times (|X(s) - Y(s)|^2 + |X(s - \tau) - \bar{Y}(s - \tau)|^2) ds \\
&\leq C \mathbb{E} \int_0^T (|X(s) - Y(s)|^{p-2} + |D(X(s - \tau)) - D(\bar{Y}(s - \tau))|^{p-2}) \\
&\quad \times (|X(s) - Y(s)|^2 + |X(s - \tau) - \bar{Y}(s - \tau)|^2) ds \\
&\leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u) + Y(u) - \bar{Y}(u)|^p) ds \\
&\quad + C \int_{-\tau}^0 \mathbb{E}(\sup_{-\tau \leq \theta \leq 0} |X(\theta) - \bar{Y}(\theta)|^p) ds \leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds + Ch^p + Ch^{p/2}.
\end{aligned} \tag{3.29}$$

In the same way as in (3.29), we can estimate $I_4(t)$ such that

$$\mathbb{E}(\sup_{0 \leq t \leq T} (I_4(t))) \leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds + Ch^p + Ch^{p/2}. \tag{3.30}$$

Using assumption **(A3)**, Hölder's inequality and (3.4), we arrive at

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} I_2(t)) &\leq \mathbb{E} \left(\int_0^T |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-1} \right. \\
&\quad \times L(1 + |\bar{Y}(s)|^l + |Y(s)|^l + 2|\bar{Y}(s - \tau)|^l)(|Y(s) - \bar{Y}(s)|) ds \Big) \\
&\leq \mathbb{E} \left(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-1} \right. \\
&\quad \times \int_0^T L(1 + |\bar{Y}(s)|^l + |Y(s)|^l + 2|\bar{Y}(s - \tau)|^l)(|Y(s) - \bar{Y}(s)|) ds \Big) \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \mathbb{E} \left[\int_0^T (1 + |\bar{Y}(s)|^l + |Y(s)|^l + |\bar{Y}(s - \tau)|^l)(|Y(s) - \bar{Y}(s)|) ds \right]^p \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \left[\int_0^T \mathbb{E}(1 + |\bar{Y}(s)|^l + |Y(s)|^l + |\bar{Y}(s - \tau)|^l)^p (|Y(s) - \bar{Y}(s)|)^p ds \right] \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \left[\int_0^T (\mathbb{E}(1 + |\bar{Y}(s)|^l + |Y(s)|^l + |\bar{Y}(s - \tau)|^l)^{2p})^{1/2} (\mathbb{E}|Y(s) - \bar{Y}(s)|^{2p})^{1/2} ds \right] \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) + Ch^{p/2}.
\end{aligned} \tag{3.31}$$

We now estimate I_3 ,

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} I_3(t)) &\leq p \mathbb{E} \int_0^T |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{p-1} \\
&\quad \times |b(\bar{Y}(s), \bar{Y}(s - \tau)) - b_h(\bar{Y}(s), \bar{Y}(s - \tau))| ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \mathbb{E} \int_0^T |b(\bar{Y}(s), \bar{Y}(s - \tau)) - b_h(\bar{Y}(s), \bar{Y}(s - \tau))|^p ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) + Ch^{\alpha p},
\end{aligned} \tag{3.32}$$

where we have used the following fact

$$\begin{aligned}
&\mathbb{E} \int_0^T |b(\bar{Y}(s), \bar{Y}(s - \tau)) - b_h(\bar{Y}(s), \bar{Y}(s - \tau))|^p ds \\
&\leq h^{\alpha p} \mathbb{E} \left[\int_0^T \frac{|b(\bar{Y}(s), \bar{Y}(s - \tau))|^{2p}}{(1 + h^\alpha |b(\bar{Y}(s), \bar{Y}(s - \tau))|)^p} ds \right] \\
&\leq h^{\alpha p} \int_0^T \mathbb{E} \left[|L(1 + |\bar{Y}(s)|^l + |\bar{Y}(s - \tau)|^l)(|\bar{Y}(s)| + |\bar{Y}(s - \tau)|) + C|^{2p} \right] ds \\
&\leq Ch^{\alpha p}.
\end{aligned} \tag{3.33}$$

Moreover, the BGD inequality yields that

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} I_5(t)) &\leq C \mathbb{E} \left(\int_0^T |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^{2p-2} \right. \\
&\quad \times \|\sigma(X(s), X(s - \tau)) - \sigma(\bar{Y}(s), \bar{Y}(s - \tau))\|^2 ds \Big)^{1/2} \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \int_0^T \mathbb{E}(|X(s) - \bar{Y}(s)|^p + |X(s - \tau) - \bar{Y}(s - \tau)|^p) ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq s \leq T} |X(s) - D(X(s - \tau)) - Y(s) + D(\bar{Y}(s - \tau))|^p) \\
&\quad + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds + Ch^p.
\end{aligned} \tag{3.34}$$

Substituting (3.29), (3.30), (3.31), (3.32) and (3.34) into (3.27).

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p) &\leq Ch^{\alpha p} + Ch^{p/2} + Ch^p + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds \\
&\leq Ch^{\alpha p} + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |X(u) - Y(u)|^p) ds.
\end{aligned} \tag{3.35}$$

The desired result follows by the Grownwall inequality and the fact $\alpha \in (0, 1/2]$. \square

In the sequel, we give a proof for Theorem 3.2 which the monotonicity condition **(A4)** is replaced by the local one **(A3)**. The techniques of the proof have been developed in Higham, Mao and Stuart [14] where they showed the strong convergence of the EM method for the SDE under a local Lipschitz condition. We therefore only give the outline of the proof here.

Proof of Theorem 3.2: For every $R > 0$, we define the stopping times

$$\tau_R := \inf\{t \geq 0 : |X(t)| \geq R\}, \quad \rho_R := \inf\{t \geq 0 : |Y(t)| \geq R\}, \quad (3.36)$$

and denote that $\theta_R = \tau_R \wedge \rho_R$, and $e(t) = X(t) - Y(t)$. By the virtue of Young's inequality, for $q > p$ and $\eta > 0$ we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t)|^p \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t)|^p I_{\{\tau_R \leq T \text{ or } \rho_R \leq T\}} \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] \\ &= \mathbb{E} \left[\left(\eta^{p/q} \sup_{0 \leq t \leq T} |e(t)|^p \right) \left(\eta^{-p/q} I_{\{\tau_R \leq T \text{ or } \rho_R \leq T\}} \right) \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] \\ &\leq \frac{p\eta}{q} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^q \right] + \frac{q-p}{q\eta^{p/(q-p)}} P(\tau_R \leq T \text{ or } \rho_R \leq T) \\ &\quad + \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] \\ &\leq \frac{p\eta}{q} 2^q \mathbb{E} \left\{ \left[\sup_{0 \leq t \leq T} |X(t)|^q \right] + \left[\sup_{0 \leq t \leq T} |Y(t)|^q \right] \right\} + \frac{q-p}{q\eta^{p/(q-p)}} \left\{ \mathbb{E} \left[\frac{|X(\tau_R)|^q}{R^q} \right] + \mathbb{E} \left[\frac{|Y(\rho_R)|^q}{R^q} \right] \right\} \\ &\quad + \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] \\ &\leq \frac{p\eta}{q} 2^q C + \frac{q-p}{q\eta^{p/(q-p)} R^p} C + \mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right]. \end{aligned}$$

In the similar way as Theorem 3.1 was proved, we can show that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] \leq C_R h + o(h). \quad (3.37)$$

Finally, given an $\epsilon > 0$, there exist some η small enough that

$$\frac{p\eta}{q} 2^q C < \frac{\epsilon}{3},$$

choose R large enough that

$$\frac{q-p}{q\eta^{p/(q-p)} R^p} 2C < \frac{\epsilon}{3},$$

and h small enough that

$$\mathbb{E} \left[\sup_{t_0 \leq t \leq T} |e(t \wedge \theta_R)|^p \right] < \frac{\epsilon}{3},$$

Hence we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right] < \epsilon.$$

□

4 Tamed EM Method of NSDDEs driven by Pure Jump Processes

In this section, we investigate NSDDEs driven by pure jump processes. Similar to NSDDEs driven by Brownian motion, most NSDDEs driven by pure jumps have no explicit solutions. Therefore, it is important to investigate the numerical approximation of NSDDEs driven by pure jumps.

We need to introduce more notation. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ -algebra on \mathbb{R} and $\lambda(dx)$ be a σ -finite measure defined on $\mathcal{B}(\mathbb{R})$. Let $D(\varrho) \subset (0, \infty)$ be a countable set, and let $\varrho = (\varrho(t), t \in D(\varrho))$ be a stationary \mathcal{F}_t -Poisson point process on \mathbb{R} with characteristic measure $\lambda(\cdot)$. Denote that by $N(dt, du)$ the Poisson counting measure associated with ϱ , i.e., $N(t, U) = \sum_{s \in D(\varrho), s \leq t} I_U(\varrho(s))$ for $U \in \mathcal{B}(\mathbb{R} \setminus \{0\})$. Let $\tilde{N}(dt, du) = N(dt, du) - dt\lambda(du)$ be the compensated Poisson measure associated with $N(dt, du)$. In what follows, we further assume that $\int_U |u|^p \lambda(du) < \infty$ for any $p \geq 1$. In this section, we consider the following NSDDE driven by pure jump processes on \mathbb{R}^n :

$$d[x(t) - G(x(t - \tau))] = f(x(t), x(t - \tau))dt + \int_U g(x(t-), x((t - \tau)-), u) \tilde{N}(du, dt), \quad t \geq 0, \quad (4.1)$$

with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in L_{\mathcal{F}_0}^p([-\tau, 0]; \mathbb{R}^n), p \geq 2$, $x(t-) := \lim_{s \uparrow t} x(s)$, where $G : \mathbb{R}^n \mapsto \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$, and $g : \mathbb{R}^n \times \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ are Borel measurable.

Again, we assume that the step size $h \in (0, 1)$ be fraction of two positive rational numbers τ and T , so that there exist two positive integers M, \bar{M} such that $h = T/M = \tau/\bar{M}$.

For the future use, we assume:

(B1) There exists a positive constant K_1 such that

$$2\langle x - G(y), f(x, y) \rangle \leq K_1(1 + |x|^2 + |y|^2) \text{ and } \int_U |g(x, y, u)|^p \lambda(du) \leq K_1(1 + |x|^p + |y|^p) \quad (4.2)$$

for $\forall x, y \in \mathbb{R}^n, p \geq 2$.

(B2) $G(0) = 0$ and there exists a constant $\kappa \in (0, 1)$ such that

$$|G(x) - G(\bar{x})| \leq \kappa|x - \bar{x}| \text{ for all } x, y \in \mathbb{R}^n. \quad (4.3)$$

(B3) For any $R > 0$, there exist two positive constants \tilde{K}_R and K_R such that for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R, p \geq 2$

$$\begin{aligned} 2\langle x - G(y) - \bar{x} + G(\bar{y}), f(x, y) - f(\bar{x}, \bar{y}) \rangle &\leq \tilde{K}_R(|x - \bar{x}|^2 + |y - \bar{y}|^2), \\ \int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^p \lambda(du) &\leq \tilde{K}_R(|x - \bar{x}|^p + |y - \bar{y}|^p), \end{aligned} \quad (4.4)$$

and

$$\sup_{|x| \leq R, |y| \leq R} |f(x, y)| \leq K_R.$$

(B4) There exist two positive constants l and L such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n, p \geq 2$

$$2\langle x - G(y) - \bar{x} + G(\bar{y}), f(x, y) - f(\bar{x}, \bar{y}) \rangle \leq L(|x - \bar{x}|^2 + |y - \bar{y}|^2)$$

$$\int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^p \lambda(du) \leq L(|x - \bar{x}|^p + |y - \bar{y}|^p)$$

and

$$|f(x, y) - f(\bar{x}, \bar{y})| \leq L(1 + |x|^l + |y|^l + |\bar{x}|^l + |\bar{y}|^l)(|x - \bar{x}| + |y - \bar{y}|).$$

(B5) For every $p > 0$, there exists a positive integer K , such that

$$\mathbb{E} \|\xi(t) - \xi(s)\|^p \leq K|t - s|^p, \text{ for any } s, t \in [-\tau, 0].$$

Remark 4.1 Under assumptions (B1), (B2) and (B4), the NSDDE (4.1) with the initial data $x(0) = \xi$ satisfying $\mathbb{E} \|\xi\|^2 < \infty$ has a pathwise unique strong solution. The details of proof can be found in [27, Theorem 170]. If the condition (B4) is replaced by the condition (B3), the existence and uniqueness theorem still holds.

Set

$$f_h(x, y) = \frac{f(x, y)}{1 + h^\alpha |f(x, y)|}, \quad \alpha \in (0, 1/2).$$

Similarly to the Brownian motion case, the discrete-time tamed EM scheme associated with (4.1) can be defined as following: For every integer $n = -\bar{M}, \dots, 0$, $z_h^{(n)} = \xi(nh)$. For every integer $n = 1, \dots, M - 1$,

$$z_h^{(n+1)} - G(z_h^{(n+1-\bar{M})}) = z_h^{(n)} - G(z_h^{(n-\bar{M})}) + f_h(z_h^{(n)}, z_h^{(n-\bar{M})})h + g(z_h^{(n)}, z_h^{(n-\bar{M})}, u)\Delta\tilde{N}_h^n, \quad (4.5)$$

where $\Delta\tilde{N}_h^n := \tilde{N}((n+1)h, U) - \tilde{N}(nh, U)$, the increment of compensated Poisson process. We may rewrite the discrete tamed EM scheme as follows:

$$z_h^{(n+1)} = G(z_h^{(n+1-\bar{M})}) + \xi(0) - G(\xi(-\tau)) + \sum_{i=1}^n f_h(z_h^{(i)}, z_h^{(i-\bar{M})})h$$

$$+ \sum_{i=1}^n g(z_h^{(i)}, z_h^{(i-\bar{M})}, u)\Delta\tilde{N}_h^i. \quad (4.6)$$

For $t \in [nh, (n+1)h)$, denote that $\bar{z}(t) := z_h^{(n)}$, and then $\bar{z}(t - \tau) = z_h^{(n-\bar{M})}$. It is more convenient to define the continuous-time tamed EM approximate solution $z(t)$ associated with (4.1) as below:

For any $\theta \in [-\tau, 0]$, $z(\theta) = \xi(\theta)$. For any $t \in [0, T]$,

$$z(t) = G(\bar{z}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t f_h(\bar{z}(s), \bar{z}(s - \tau))ds$$

$$+ \int_0^t \int_U g(\bar{z}(s-), \bar{z}((s - \tau)-), u)\tilde{N}(du, ds). \quad (4.7)$$

Since for any $t > 0$ there exists a positive integer n , $0 \leq n \leq M - 1$, such that $t \in [nh, (n+1)h)$, we have

$$z(t) = z(nh) + \int_{nh}^t f_h(\bar{z}(s), \bar{z}(s - \tau))ds + \int_{nh}^t \int_U g(\bar{z}(s-), \bar{z}((s - \tau)-), u)\tilde{N}(du, ds). \quad (4.8)$$

Clearly, continuous-time tamed EM approximate solution $z(t)$ coincides with the discrete-time tamed approximation solution $\bar{z}(t)$ at the grid points $t = nh$, i.e. $\bar{z}(t) = z_h^n = z(nh)$.

We now state our main results of this section.

Theorem 4.1 *Assume that (B1), (B2), (B4) and (B5) hold. Assume also that $p \geq 2$ and $\alpha \in (0, 1/p)$, then the tamed EM scheme (4.7) converges to the exact solution of (4.1) such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |z(t) - x(t)|^p \right] \leq Ch^\gamma, \quad (4.9)$$

where $\gamma = 1/2 \wedge \alpha p$.

and

Theorem 4.2 *Assume that (B1)-(B3) and (B5) hold. Assume also that $p \geq 2$ and $\alpha \in (0, 1/p)$, then the tamed EM scheme (4.7) converges to the exact solution of (4.1) such that*

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} |z(t) - x(t)|^p \right] = 0. \quad (4.10)$$

4.1 Boundedness of Moments

In order to show our main results, we need the following inequality [21, Theorem 1].

Lemma 4.1 *Let $\varphi : \mathbb{R}_+ \times \rightarrow \mathbb{R}^n$ and assume that*

$$\int_0^t \int_U \mathbb{E} |\varphi(s, u)|^2 \lambda(du) ds < \infty, \quad t \geq 0 \quad a.s..$$

Then there exist a constant $C_p > 0$ such that for any $p > 0$

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq t} \left| \int_0^s \int_U \varphi(r-, u) \tilde{N}(dr, du) \right|^p \right) &\leq C_p \left[\mathbb{E} \left(\int_0^t \int_U |\varphi(s, u)|^2 \lambda(du) ds \right)^{p/2} \right. \\ &\quad \left. + \mathbb{E} \int_0^t \int_U |\varphi(s, u)|^p \lambda(du) ds \right]. \end{aligned}$$

It is known that if $1 \leq p \leq 2$, then the second term on the right hand side can be eliminated.

Lemma 4.2 *Consider the continuous-time tamed EM scheme given by equation (4.8). Assume that $p \geq 2, \alpha \in (0, 1/p)$, and*

$$\sup_{0 \leq t \leq T} \mathbb{E}(|z(t)|^p) \leq C. \quad (4.11)$$

Assume also that (B1) holds, then it holds that

$$\mathbb{E} \left[\sup_{0 \leq n \leq M-1} \sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p \right] \leq Ch, \quad (4.12)$$

and

$$\mathbb{E} \left[\sup_{0 \leq n \leq M-1} \sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p |f_h(\bar{z}(t), \bar{z}(t - \tau))|^p \right] \leq C. \quad (4.13)$$

Proof: By the definition of $z(t)$, we can write that

$$\begin{aligned} \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p \right] &= \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t f_h(\bar{z}(s), \bar{z}(s-\tau)) ds \right. \right. \\ &\quad \left. \left. + \int_{nh}^t \int_U g(\bar{z}(s-), \bar{z}((s-\tau)-), u) \tilde{N}(du, ds) \right|^p \right]. \end{aligned}$$

Therefore, due to Hölder's inequality,

$$\begin{aligned} &\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p \right] \\ &\leq 2^{p-1} h^{p-1} \mathbb{E} \left[\int_{nh}^{(n+1)h} |f_h(\bar{z}(s), \bar{z}(s-\tau))|^p ds \right] \\ &\quad + 2^{p-1} \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t \int_U g(\bar{z}(s-), \bar{z}((s-\tau)-), u) \tilde{N}(du, ds) \right|^p \right]. \end{aligned} \tag{4.14}$$

Using the Lemma 4.1 and (4.11), for some $p \geq 2$ we have

$$\begin{aligned} &\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} \left| \int_{nh}^t \int_U g(\bar{z}(s-), \bar{z}((s-\tau)-), u) \tilde{N}(du, ds) \right|^p \right] \\ &\leq C \mathbb{E} \int_{nh}^{(n+1)h} \int_U |g(\bar{z}(s), \bar{z}(s-\tau), u)|^p \lambda(du) ds \\ &\leq C \mathbb{E} \int_{nh}^{(n+1)h} (1 + |\bar{z}(s)|^p + |\bar{z}(s-\tau)|^p) ds \\ &\leq Ch. \end{aligned}$$

This, together with $|f_h(\bar{z}(s), \bar{z}(s-\tau), s)| \leq h^{-\alpha}$, yields

$$\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p \right] \leq 2^{p-1} h^{(1-\alpha)p} + Ch \leq Ch. \tag{4.15}$$

Hence (4.12) holds. Moreover,

$$\begin{aligned} &\mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p |f_h(\bar{z}(s), \bar{z}(s-\tau))|^p \right] \\ &\leq \mathbb{E} \left[\sup_{nh \leq t \leq (n+1)h} |z(t) - z(nh)|^p \right] h^{-\alpha p} \leq Ch^{1-\alpha p} \leq C, \end{aligned} \tag{4.16}$$

as required. The proof is therefore complete. \square

The results of boundedness are given below:

Lemma 4.3 *Assume that (B1) and (B2) hold. Assume also $p \geq 2, \alpha \in (0, 1/p)$, then there exists a constant C such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |x(t)|^p \right] \vee \mathbb{E} \left[\sup_{0 \leq t \leq T} |z(t)|^p \right] \leq C. \tag{4.17}$$

Proof: An application of the Itô formula yields,

$$\begin{aligned}
& |x(t) - G(x(t - \tau))|^p = |\xi(0) - G(\xi(-\tau))|^p \\
& + p \int_0^t |x(s) - G(x(s - \tau))|^{p-2} \langle x(s) - G(x(s - \tau)), f(x(s), x(s - \tau)) \rangle ds \\
& + \int_0^t \int_U |x(s) - G(x(s - \tau)) + g(x(s), x(s - \tau), u)|^p - |x(s) - G(x(s - \tau))|^p \\
& - p |x(s) - G(x(s - \tau))|^{p-2} \langle x(s) - G(x(s - \tau)), g(x(s), x(s - \tau), u) \rangle \lambda(du) ds \\
& + \int_0^t \int_U p |x(s) - G(x(s - \tau))|^{p-2} \\
& \times \langle x(s) - G(x(s - \tau)), g(x(s), x(s - \tau), u) \rangle \tilde{N}(du, ds) \\
& =: J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}$$

Due to assumption **(B1)**, we derive that

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} J_2(t)) \\
& \leq C \mathbb{E} \int_0^T |x(s) - G(x(s - \tau))|^{p-2} (1 + |x(s)|^2 + |x(s - \tau)|^2) ds \\
& \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |x(u)|^p) ds.
\end{aligned} \tag{4.18}$$

By the Taylor expansion, we have

$$|x + y|^p - |x|^p - |x|^{p-2}|xy| \leq C(|x|^{p-2}|y|^2 + |y|^p). \tag{4.19}$$

This, together with **(B1)** and **(B2)**, implies

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} J_3(t)) = \mathbb{E} \int_0^T \int_U |x(s) - G(x(s - \tau)) + g(x(s), x(s - \tau), u)|^p \\
& - |x(s) - G(x(s - \tau))|^p - p |x(s) - G(x(s - \tau))|^{p-2} \\
& \times \langle x(s) - G(x(s - \tau)), g(x(s), x(s - \tau), u) \rangle \lambda(du) ds \\
& \leq \mathbb{E} \int_0^T \int_U [|x(s) - G(x(s - \tau))|^{p-2} |g(x(s), x(s - \tau), u)|^2 \\
& + |g(x(s), x(s - \tau), u)|^p] \lambda(du) ds \\
& \leq \mathbb{E} \int_0^T |x(s) - G(x(s - \tau))|^{p-2} (1 + |x(s)|^2 + |x(s - \tau)|^2) \\
& + (1 + |x(s)|^p + |x(s - \tau)|^p) ds \\
& \leq C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |x(u)|^p) ds.
\end{aligned} \tag{4.20}$$

Using Lemma 4.1 and Young's inequality, one has

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} J_4(t)) \\
& \leq C \mathbb{E} \left(\int_0^T \int_U |x(s) - G(x(s - \tau))|^{2p-2} |g(x(s), x(s - \tau), u)|^2 \lambda(du) ds \right)^{1/2} \\
& \leq C \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |x(t) - G(x(s - \tau))|^{p-1} \right) \times \left(\int_0^T \int_U |g(x(s), x(s - \tau), u)|^2 \right)^{1/2} \right] \quad (4.21) \\
& \leq \frac{1}{4} \mathbb{E} \left(\sup_{0 \leq t \leq T} |x(t) - G(x(s - \tau))|^p \right) + C \mathbb{E} \left(\int_0^T (1 + |x(s)|^2 + |x(s - \tau)|^2) \right)^{p/2} \\
& \leq \frac{1}{4} \mathbb{E} \left(\sup_{0 \leq t \leq T} |x(t) - G(x(s - \tau))|^p \right) + C + C \int_0^T \mathbb{E} \left(\sup_{0 \leq u \leq s} |x(u)|^p \right) ds.
\end{aligned}$$

Applying [20, Lemma 6.4.4] (also see (3.10)), we derive that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C + C \int_0^T \mathbb{E} \left(\sup_{0 \leq u \leq s} |x(u)|^p \right) ds.$$

By the Gronwall inequality, we have

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C.$$

The proof of boundedness of EM approximation is analogous to its Brownian motion counterpart, we first claim that there exists a constant $C > 0$ such that:

$$\sup_{0 \leq t \leq T} \mathbb{E}(|z(t)|^2) \leq C. \quad (4.22)$$

Using the Itô formula, we have

$$\begin{aligned}
& |z(t) - G(\bar{z}(t - \tau))|^2 = |\xi(0) - G(\xi(-\tau))|^2 \\
& + 2 \int_0^t \langle \bar{z}(s) - G(\bar{z}(s - \tau)), f_h(\bar{z}(s), \bar{z}(s - \tau)) \rangle ds \\
& + 2 \int_0^t \langle z(s) - \bar{z}, f_h(\bar{z}(s), \bar{z}(s - \tau)) \rangle ds \\
& + \left(\int_0^t \int_U |z(s) - G(\bar{z}(s - \tau)) + g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 - |z(s) - G(\bar{z}(s - \tau))|^2 \right. \quad (4.23) \\
& \left. - 2 \langle z(s) - G(\bar{z}(s - \tau)), g(\bar{z}(s), \bar{z}(s - \tau), u) \rangle \right) \lambda(du) ds \\
& + \int_0^t \int_U 2 \langle z(s-) - G(\bar{z}((s - \tau)-)), g(\bar{z}(s-), \bar{z}((s - \tau)-), u) \rangle \tilde{N}(du, ds) \\
& =: |\xi(0) - G(\xi(-\tau))|^2 + \bar{J}_1(t) + \bar{J}_2(t) + \bar{J}_3(t) + \bar{J}_4(t).
\end{aligned}$$

By **(B1)**, we compute

$$\begin{aligned}
\mathbb{E}(\bar{J}_1(t)) & \leq \mathbb{E} \int_0^t C(1 + |\bar{z}(s)|^2 + |\bar{z}(s - \tau)|^2) ds \\
& \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}(|z(u)|^2) ds. \quad (4.24)
\end{aligned}$$

Using the definition of $z(s)$ and $\bar{z}(s)$, we have

$$\begin{aligned}
\mathbb{E}(\bar{J}_2(t)) &= 2\mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s f_h(\bar{z}(r), \bar{z}(r-\tau)) dr, f_h(\bar{z}(s), \bar{z}(s-\tau)) \right\rangle ds \\
&+ 2\mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s g_h(\bar{z}(r-), \bar{z}((r-\tau)-), u) \tilde{N}(du, dr), f_h(\bar{z}(s), \bar{z}(s-\tau)) \right\rangle ds \\
&= 2\mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s f_h(\bar{z}(r), \bar{z}(r-\tau)) dr, f_h(\bar{z}(s), \bar{z}(s-\tau)) \right\rangle ds \\
&+ 2\mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s g_h(\bar{z}(r-), \bar{z}((r-\tau)-), u) \tilde{N}(du, dr) \Big| \mathcal{F}_{[\frac{s}{h}]h}, f_h(\bar{z}(s), \bar{z}(s-\tau)) \right\rangle ds \\
&= 2\mathbb{E} \int_0^t \left\langle \int_{[\frac{s}{h}]h}^s f_h(\bar{z}(r), \bar{z}(r-\tau)) dr, f_h(\bar{z}(s), \bar{z}(s-\tau)) \right\rangle ds \\
&\leq cth^{1-2\alpha} \leq C.
\end{aligned} \tag{4.25}$$

By using (4.19), we have

$$\begin{aligned}
\mathbb{E}(\bar{J}_3(t)) &\leq C\mathbb{E} \int_0^t (1 + |\bar{z}(s)|^2 + |\bar{z}(s-\tau)|^2) ds \\
&\leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}(|z(u)|^2) ds.
\end{aligned} \tag{4.26}$$

By taking the expectation of $\bar{J}_4(t)$, we can write that $\mathbb{E}(\bar{J}_4(t)) = 0$. Now recall (3.19), we have

$$\begin{aligned}
\sup_{0 \leq t \leq T} \mathbb{E}(|z(t)|^2) &\leq C + C \left(\sup_{0 \leq t \leq T} \mathbb{E}(|z(t) - D(\bar{z}(t-\tau))|^2) \right) \\
&\leq C + C \int_0^T \sup_{0 \leq u \leq s} \mathbb{E}(|z(u)|^2) ds.
\end{aligned}$$

The required result (4.22) will follow an application of the Gronwall inequality. Now, letting $p = 4$ and using the Itô formula, we have

$$\begin{aligned}
|z(t) - G(\bar{z}(t-\tau))|^p &= |\xi(0) - G(\xi(-\tau))|^p \\
&+ p \int_0^t |z(s) - G(\bar{z}(s-\tau))|^{p-2} \langle z(s) - G(\bar{z}(s-\tau)), f_h(\bar{z}(s), \bar{z}(s-\tau)) \rangle ds \\
&+ \left(\int_0^t \int_U |z(s) - G(\bar{z}(s-\tau)) + g(\bar{z}(s), \bar{z}(s-\tau), u)|^p - |z(s) - G(\bar{z}(s-\tau))|^p \right. \\
&\quad \left. - p|z(s) - G(\bar{z}(s-\tau))|^{p-2} \langle z(s) - G(\bar{z}(s-\tau)), g(\bar{z}(s), \bar{z}(s-\tau), u) \rangle \right) \lambda(du) ds \\
&+ \int_0^t \int_U p|z(s-) - G(\bar{z}((s-\tau)-))|^{p-2} \langle z(s-) - G(\bar{z}((s-\tau)-)), \\
&\quad g(\bar{z}(s-), \bar{z}((s-\tau)-), u) \rangle \tilde{N}(du, ds) \\
&=: F_1(t) + F_2(t) + F_3(t) + F_4(t).
\end{aligned} \tag{4.27}$$

Noting that

$$\begin{aligned}
F_2(t) &= p \int_0^t |z(s) - G(\bar{z}(s-\tau))|^{p-2} \langle \bar{z}(s) - G(\bar{z}(s-\tau)), f_h(\bar{z}(s), \bar{z}(s-\tau)) \rangle ds \\
&+ p \int_0^t |z(s) - G(\bar{z}(s-\tau))|^{p-2} \langle z(s) - \bar{z}(s), f_h(\bar{z}(s), \bar{z}(s-\tau)) \rangle ds,
\end{aligned}$$

using assumption **(B1)**, (4.22) and (4.13), we arrive at

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} F_2(t)) &\leq C \mathbb{E} \int_0^T |z(s) - G(\bar{z}(s - \tau))|^{p-2} (1 + |\bar{z}(s)|^2 + |\bar{z}(s - \tau)|^2) ds \\
&+ \mathbb{E}(\sup_{0 \leq t \leq T} \int_0^T |z(s) - G(\bar{z}(s - \tau))|^{p-2} |(z(s) - \bar{z}(s)) f_h(\bar{z}(s), \bar{z}(s - \tau))| ds) \\
&\leq C \mathbb{E} \int_0^T |z(s) - G(\bar{z}(s - \tau))|^{p-2} (1 + |\bar{z}(s)|^2 + |\bar{z}(s - \tau)|^2) ds \\
&+ \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |z(u) - \bar{z}(u)|^{p/2} |f_h(\bar{z}(u), \bar{z}(u - \tau))|^{p/2}) ds \\
&\leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |z(u)|^p) ds + \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + C.
\end{aligned} \tag{4.28}$$

By (4.19), **(B1)** and **(B2)**, we obtain

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} F_3(t)) &= \mathbb{E} \int_0^T \int_U |z(s) - G(\bar{z}(s - \tau)) + g(\bar{z}(s), \bar{z}(s - \tau), u)|^p - |z(s) - G(\bar{z}(s - \tau))|^p \\
&- p |z(s) - G(\bar{z}(s - \tau))|^{p-2} \langle z(s) - \bar{z}(s - \tau), g(\bar{z}(s), \bar{z}(s - \tau), u) \rangle \lambda(du) ds \\
&\leq \mathbb{E} \int_0^T \int_U |z(s) - G(\bar{z}(s - \tau))|^{p-2} |g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 + |g(\bar{z}(s), \bar{z}(s - \tau), u)|^p \lambda(du) ds \\
&\leq \mathbb{E} \int_0^T |z(s) - G(\bar{z}(s - \tau))|^{p-2} (1 + |\bar{z}(s)|^2 + |\bar{z}(s - \tau)|^2) + (1 + |\bar{z}(s)|^p + |\bar{z}(s - \tau)|^p) ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + \mathbb{E} \int_0^T 1 + |\bar{z}(s)|^p + |\bar{z}(s - \tau)|^p ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |z(u)|^p) ds.
\end{aligned} \tag{4.29}$$

Using Lemma 4.1 and Young's inequality, we derive that

$$\begin{aligned}
&\mathbb{E}(\sup_{0 \leq t \leq T} F_4(t)) \\
&\leq C \mathbb{E} \left(\int_0^T \int_U |z(s) - G(\bar{z}(s - \tau))|^{2p-2} \times |g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 \lambda(du) ds \right)^{1/2} \\
&\leq C \mathbb{E} [(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^{p-1}) \times (\int_0^T \int_U |g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 \lambda(du) ds)^{1/2}] \\
&\leq C \mathbb{E} \int_0^T |z(s) - G(\bar{z}(s - \tau))|^{p-2} (1 + |\bar{z}(s)|^2 + |\bar{z}(s - \tau)|^2) ds \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + C \mathbb{E}(\int_0^T 1 + |z(s)|^p + |\bar{z}(s)|^p + |\bar{z}(s - \tau)|^p ds) \\
&\leq \frac{1}{4} \mathbb{E}(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p) + C + C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |z(u)|^p) ds.
\end{aligned} \tag{4.30}$$

Now substituting (4.28), (4.29) and (4.30) into (4.27), we obtain that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z(t) - G(\bar{z}(t - \tau))|^p\right) \leq C + C \int_0^T \mathbb{E}\left(\sup_{0 \leq u \leq s} |z(u)|^p\right) ds.$$

By applying [20, Lemma 6.4.4] (also see (3.10)), we derive that

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z(t)|^p\right) \leq C + C \int_0^T \mathbb{E}\left(\sup_{0 \leq u \leq s} |z(u)|^p\right) ds.$$

By the Gronwall inequality, we have for $p = 4$,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z(t)|^p\right) \leq C.$$

By repeating the same procedure, the desired result (4.17) can be obtained by induction. \square

4.2 Proof of the Main Results

In this subsection, we shall prove our main results.

Proof of Theorem 4.1: Recall that

$$\begin{aligned} |x(t) - z(t)|^p &\leq \kappa |x(t - \tau) - \bar{z}(t - \tau)|^p \\ &\leq \kappa |x(t) - \bar{z}(t)|^p I_{[-\tau \leq t \leq 0]}(t) + \kappa |x(t) - \bar{z}(t)|^p I_{[0 \leq t \leq T]}(t) \\ &\quad + \frac{1}{(1 - \kappa)^{p-1}} |x(t) - G(x(t - \tau)) - z(t) + G(\bar{z}(t - \tau))|^p, \end{aligned} \tag{4.31}$$

which implies that

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} |x(t) - z(t)|^p\right) &\leq \frac{1}{(1 - \kappa)^p} \mathbb{E}\left(\sup_{0 \leq t \leq T} |x(t) - G(x(t - \tau)) \right. \\ &\quad \left. - z(t) + G(\bar{z}(t - \tau))|^p\right) + Ch^p. \end{aligned} \tag{4.32}$$

An application of the Itô formula yields that for any $p \geq 2$,

$$\begin{aligned}
& |x(t) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau))|^p \\
&= p \int_0^t |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times \langle x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau)), \\
&\quad \quad f(x(s), x(s - \tau)) - f_h(\bar{z}(s), \bar{z}(s - \tau)) \rangle ds \\
&\quad + \int_0^t \int_U |x(s) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau)) \\
&\quad \quad + (g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u))|^p \\
&\quad - |x(s) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau))|^p \\
&\quad \quad - p|x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times \langle x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau)), \\
&\quad \quad g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u) \rangle \lambda(du) ds \\
&\quad + \int_0^t \int_U |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times \langle x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau)), \\
&\quad \quad g(x(s-), x((s - \tau)-), u) - g(\bar{z}(s-), \bar{z}((s - \tau)-), u) \rangle \tilde{N}(du, ds) \\
&=: \bar{F}_1(t) + \bar{F}_2(t) + \bar{F}_3(t)
\end{aligned} \tag{4.33}$$

Using the similar way as in the (3.29), (3.31) and (3.32) , we derive that

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \bar{F}_1(t)) \leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |x(u) - z(u)|^p) ds + C \int_{-\tau}^0 \mathbb{E}(\sup_{-\tau \leq \theta \leq 0} |x(\theta) - \bar{z}(\theta)|^p) ds \\
& + \frac{1}{2} \mathbb{E}(\sup_{0 \leq s \leq T} |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^p) \\
& + C[(\int_0^T \mathbb{E}(1 + \sup_{0 \leq u \leq s} |z(s)|^l)^{2p} ds)]^{1/2} [\int_0^T \mathbb{E}(|z(s) - \bar{z}(s)|^{2p}) ds]^{1/2} \\
& + C \int_0^T \mathbb{E}|f(\bar{z}(u), \bar{z}(u - \tau)) - f_h(\bar{z}(u), \bar{z}(u - \tau))|^p ds \\
& \leq C \int_0^T \mathbb{E}(\sup_{0 \leq u \leq s} |x(u) - z(u)|^p) ds + Ch^p + Ch^{1/2} + Ch^{\alpha p} \\
& + \frac{1}{2} \mathbb{E}(\sup_{0 \leq s \leq T} |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^p) \\
& \leq C \int_0^t \mathbb{E}[\sup_{0 \leq u \leq s} |x(u) - z(u)|^p] ds \\
& + \frac{1}{2} \mathbb{E}(\sup_{0 \leq s \leq t} |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^p) + Ch^\gamma,
\end{aligned} \tag{4.34}$$

where $\gamma = 1/2 \wedge \alpha p$.

By **(B3)**, **(B4)** and (4.19), we compute

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \bar{F}_2(t)) \\
&= \mathbb{E} \int_0^t \int_U |x(s) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau)) \\
&\quad + (g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u))|^p \\
&\quad - |x(s) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau))|^p \\
&\quad - p|x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times \langle x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau)), \\
&\quad g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u) \rangle \lambda(du) ds \\
&\leq \mathbb{E} \int_0^T \int_U |x(s) - G(x(s - \tau)) - z(t) + G(\bar{z}(t - \tau))|^{p-2} \\
&\quad \times |g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 \\
&\quad + |g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u)|^p \lambda(du) ds \\
&\leq \mathbb{E} \int_0^T |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times (|x(s) - z(s)|^2 + |x((s - \tau) - \bar{z}(s - \tau)|^2) ds \\
&\quad + \int_0^T |x(s) - z(s)|^p + |x(s - \tau) - \bar{z}(s - \tau)|^p ds \\
&\leq Ch^p + C \int_0^t \mathbb{E}[\sup_{0 \leq u \leq s} |x(u) - z(u)|^p] ds.
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \bar{F}_3(t)) \\
&\leq C \mathbb{E} \left(\int_0^T \int_U |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{2p-2} \right. \\
&\quad \times |g(x(s), x(s - \tau), u) - g(\bar{z}(s), \bar{z}(s - \tau), u)|^2 \lambda(du) ds \Big) \\
&\leq C \mathbb{E} \int_0^T |x(s) - G(x(s - \tau)) - z(s) + G(\bar{z}(s - \tau))|^{p-2} \\
&\quad \times (|x(s) - z(s)|^2 + |x(s - \tau) - \bar{z}(s - \tau)|^2) ds \\
&\leq Ch^p + C \int_0^t \mathbb{E}[\sup_{0 \leq u \leq s} |x(u) - z(u)|^p] ds.
\end{aligned} \tag{4.36}$$

Now, substituting (4.34), (4.35) and (4.36) into (4.33), and using (4.32), we have

$$\begin{aligned}
\mathbb{E}(\sup_{0 \leq t \leq T} |x(t) - z(t)|^p) &\leq Ch^\gamma + Ch^p + C \mathbb{E} \int_0^t \sup_{0 \leq u \leq s} |x(u) - z(u)|^p ds \\
&\leq Ch^\gamma + C \int_0^t \mathbb{E}[\sup_{0 \leq u \leq s} |x(u) - z(u)|^p] ds.
\end{aligned} \tag{4.37}$$

An application of the Grownwall inequality yields the desired result. \square

Proof of Theorem 4.2: Noting the fact that all sample paths associated with (4.1) are discontinuous, we define the following stopping times: For every $R > 0$, stopping times are defined as:

$$\tau_R := \inf\{t \geq 0 : |x(t)| > R\}, \quad \rho_R := \inf\{t \geq 0 : |z(t)| > R\}. \quad (4.38)$$

The remainder of the proof is similar to the that of Theorem 3.2, we omit details here. \square

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